Matrici și determinanți

Matrice de tipul $m \times n$ (m-linii și n-coloane) cu coeficienți în K, unde K este $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (sau chiar $\mathbb{Z}$, $\mathbb{Z}_n$):

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$a_{ij} \in K, \forall i, j.$

Operații

- adunarea: $(a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} + (b_{ij})_{\dots} = (c_{ij})_{\dots}$, $c_{ij} = a_{ij} + b_{ij}$ (de același tip)

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x+\alpha & y+\beta \\ z+\gamma & t+\delta \end{pmatrix}$$

- înmulțirea: $(a_{ij}) \cdot (b_{ij}) = (d_{ij})$ – nr de coloane pt (a_{ij}) = nr de linii din (b_{ij})

$$d_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{ip} \cdot b_{pj}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix} = \begin{pmatrix} \_ & \_ & d_{13} & \dots \end{pmatrix}$$

$$d_{13} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} b_{33}$$

! Înmulțirea matricilor nu este comutativă:

dacă $\exists$ AB și BA nu știm că $AB = BA$.

1. Fie X o matrice cu coeficienți reali a.î.:

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} X = X \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

A) 26
B) 51
C) 21
D) 2

$(x^2 = 1 \not\Rightarrow x = 1 \Rightarrow x = \pm 1)$

Alegeți variantele corecte din cele de mai jos:

A) $X = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$; B) $X = \begin{pmatrix} \alpha & \beta \\ \frac{3\beta}{2} & \alpha \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$

C) $X = \begin{pmatrix} a & 2b \\ 3b & a \end{pmatrix}$, $a, b \in \mathbb{R}$

$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ $m = q$, $2p = 3n$; $n = \beta \Rightarrow p = \frac{3\beta}{2}$; $n = 2b \Rightarrow p = 3b$

! D) Orice matrice din $\mathcal{M}_2(\mathbb{R})$ verifică egalitatea de mai sus.

Soluție: $\exists \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} X \Rightarrow X$ are 2 linii
$\exists X \cdot \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow X$ are 2 coloane
$\Rightarrow X \in \mathcal{M}_2(\mathbb{R})$

$$X = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} m + 2p & n + 2q \\ 3m + p & 3n + q \end{pmatrix} = \begin{pmatrix} m + 3n & 2m + n \\ p + 3q & 2p + q \end{pmatrix}$$

$$\begin{cases} 2p = 3n \\ 2q = 2m \\ 3m = 3q \\ 3n = 2p \end{cases} \Leftrightarrow \begin{cases} 2p = 3n \\ m = q \end{cases}, \quad m, n, p, q \in \mathbb{R}.$$

2. Fie $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ și $p, q \in \mathbb{R}$ a.î.

$$A^{2021} = p \cdot A^2 + q A.$$

A) $pq < 0$ B) p par și q impar ($\in \mathbb{Z}$)

C) $5 \mid p$ D) $3 \nmid p$.

Soluție

$$A^2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix},$$

$$A^n = \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & 1 & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix} \overset{?}{\Longrightarrow} A^{n+1} = \begin{pmatrix} 2^n & 0 & 2^n \\ 0 & 1 & 0 \\ 2^n & 0 & 2^n \end{pmatrix}$$

$$A^{n+1} = A^n \cdot A = \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & 1 & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n-1}+2^{n-1} & 0 & \\ & & \dots \end{pmatrix}$$

$$2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$$

$\Rightarrow$ formula pt A^n e corectă

$$\Rightarrow A^{2021} = \begin{pmatrix} 2^{2020} & 0 & 2^{2020} \\ 0 & 1 & 0 \\ 2^{2020} & 0 & 2^{2020} \end{pmatrix} = p \cdot \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} + q \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} 2^{2020} = p\cdot 2 + q \\ 1 = p + q \end{cases} \Rightarrow \begin{cases} p = 2^{2020} - 1 > 0 \\ q = 2 - 2^{2020} < 0 \end{cases}$$

$p \cdot q < 0$ $\underline{\underline{\text{ok}}}$ A ✓

p - impar ~~B~~

$$2^{2020} - 1 = (2^{1010} - 1)(2^{1010} + 1) =$$
$$= (2^{505} - 1)(2^{505} + 1)(2^{1010} + 1) \ldots$$

$$2^{2020} - 1 = (2^4)^{505} - 1 = (2^4 - 1)\cdot(\ \ldots\ )$$

k - impar $\Rightarrow$ $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + b^{k-1})$

$\Rightarrow$ $2^4 - 1 = 15 \mid 2^{2020} - 1 = p \Rightarrow 5, 3 \mid p$

C ✓ ~~D~~.

X - matrice pătratică $\overset{def}{\Longrightarrow}$ $X^0 = I_n$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A \in \mathcal{M}_n(K) \longmapsto \det(A) \in K$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$ → putem scoate „minori"

linia i_1 →
linia i_2 →
linia i_k

$$\begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \ldots & a_{i_1 j_k} \\ a_{i_2 j_1} & a_{i_2 j_2} & \ldots & \\ \ldots & - & - & \end{vmatrix}$$

↑ col j_1, col j_2 ...

$\underline{\text{Ex}}$ $$\begin{vmatrix} a_{3,5} & a_{3,7} & a_{3,10} \\ a_{8,5} & a_{8,7} & a_{8,10} \\ a_{10,5} & a_{10,7} & a_{10,10} \end{vmatrix}$$.

rang A = cea mai mare dimensiune a unui minor nenul al lui A.

pt calcul: se pleacă de la minori mici nenuli care se bordează ...

3. $A = \begin{pmatrix} 1 & -2 & -2 \\ 3 & 1 & a \\ 3 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 & -2 & 4 \\ 3 & 1 & a & 4 \\ 3 & -1 & 1 & b \end{pmatrix}$ $a, b \in \mathbb{R}$.

Dacă A și B au ambele rangul 2, atunci

(A) $\det(A) = \det(B) = 0$

(B) $a > b$

(C) $a < 0$, $b > 0$

(D) $a > 0$, $b > 0$.

A - NU E CORECT !!!

$\nexists \det(B)$

$\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7 \neq 0$

$\operatorname{rang} A = 2 \Rightarrow \begin{vmatrix} 1 & -2 & -2 \\ 3 & 1 & a \\ 3 & -1 & 1 \end{vmatrix} = 0 \Rightarrow a = 19/5$

rang B = 2 ⇒ ——— " ———

$\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 4 \\ 3 & -1 & b \end{vmatrix} = 0 \Rightarrow b = \frac{44}{7}$

Temă: 1) Calculați A^n pt:

a) $A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$

2.

$$D(a,b) = \begin{vmatrix} 1 & a-b & a^2+b^2-2ab \\ 5 & 5a-3b & 5a^2+b^2-6ab \\ 10 & 10a-2b & 10a^2-2b^2-4ab \end{vmatrix}$$

(A) $D(a,b)$ nu depinde de a.

(B) $\forall a \in \mathbb{R}$, $f_a : \mathbb{R} \to \mathbb{R}$, $f_a(x) = D(a,x)$ este impară

(C) $\forall a,b \in \mathbb{R}$ $D(a,b) = -D(b,a)$

(D) $D(2,5) = 1000$.

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